Reply to "Comment on 'Invariance principle for extension of hydrodynamics: Nonlinear viscosity' "

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A relevant analysis of the invariance equation of Karlin, Dukek, and Nonnenmacher [Phys. Rev. E **55**, 1573 (1997)] demonstrates that the dynamic viscosity factor exists for all values of the longitudinal rate, both positive and negative. We give an explanation of difficulties of the numerical method used by Uribe and Piña, and suggest an alternative approach. [S1063-651X(98)07503-5]

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In Ref. [1], we derived a correction to the Navier-Stokes expression for the stress σ, in the one-dimensional case, for large values of the average velocity u. This correction has the form $\sigma=-\mu R(g)\partial_x u$, where $g\propto\partial_x u$ is the longitudinal rate. The viscosity factor $R(g)$ is a solution of a differential equation, subject to a certain initial condition [Eq. (1) of Ref. [2]]. This equation was not studied completely in Ref. [1]. In their Comment, Uribe and Piña indicated some interesting features of this equation. In particular, they asked what happens to the relevant solution at negative values of g?

Let us denote the points of the (g,R) plane as $P=(g,R)$. The relevant solution $R(g)$ emerges from point $P_0=(0,\frac{4}{3})$, and, as we demonstrate below, it can be unequally continued to arbitrary values of g, positive and negative. This solution can be constructed, for example, with the Taylor expansion used in Ref. [2], and which is identical to the relevant subseries of the Chapman-Enskog expansion [cf. Ref. [1], Eq. (8)]. However, the difficulty in constructing this solution numerically for $g<0$ originates from the fact that the same point P_0 is the point of *essential singularity* of other (irrelevant) solutions to Eq. (1). Indeed, for $|g|\ll 1$, let us consider $\tilde{R}(g)=R(g)+\Delta$, where $R(g)=\frac{4}{3}+\frac{8}{9}(\gamma-2)g$ is the relevant solution for small $|g|$, and $\Delta(g)$ is a deviation. Neglecting in Eq. (1) all regular terms (of the order g^2), and also neglecting $g\Delta$ in comparison to Δ, we derive the following equation: $(1-\gamma)g^2(d\Delta/dg)=-\frac{3}{2}\Delta$. The solution is $\Delta(g)=\Delta(g_0)\exp[a(g^{-1}-g_0^{-1})]$, where $a=(3/2)(1-\gamma)^{-1}$. The essential singularity at $g=0$ is apparent from this solution, unless $\Delta(g_0)\neq 0$ (that is, no singularity exists except for the relevant solution $\tilde{R}=R$). Let $\Delta(g_0)\neq 0$. If $g<0$, then $\Delta\to 0$, together with all its derivatives, as $g\to 0$. If $g>0$, the solution expands, as $g\to 0$.

The complete picture for $\gamma\neq 1$ is as follows: The lines $g=0$ and $P=(g,g^{-1})$ define the boundaries of the domain of attraction $A=A_-\cup A_+$, where $A_-=\{P|-\infty<g<0,R>g^{-1}\}$, and $A_+=\{P|\infty>g>0,R<g^{-1}\}$. The graph $G=(g,R(g))$ of the relevant solution belongs to the closure of A, and goes through the points $P_0=(0,\frac{4}{3})$, $P_-=(-\infty,0)$, and $P_+=(\infty,0)$. These points at the boundaries of A are the points of essential singularity of any other (irrelevant) solution with the initial conditions $P\in A$, and $P\notin A\cap G$. That is, if $P\in A_+$, $P\notin A_+\cap G$, the solution expands at P_0, and is attracted to P_+. If $P\in A_-$, and $P\notin A_-\cap G$, the solution expands at P_-, and is attracted to P_0. It is this latter case that was found numerically by Uribe and Piña.

The above consideration is supported by our independent numerical study of Eq. (1) (see Fig. 1, corresponding to the case of hard spheres, $\gamma=\frac{1}{2}$). The difficulty of numerical integration from P_0 to negative values of g is quite clear: the integration then goes in a direction opposite to the direction of attraction of the irrelevant solutions. However, the same feature becomes an advantage for the integration to the positive values of g: because of the attraction of all irrelevant solutions to the relevant one, roundoff errors will be suppressed. This explains why no difficulty was encountered in this part of integration in Ref. [2].

One can utilize the attraction in the negative domain in

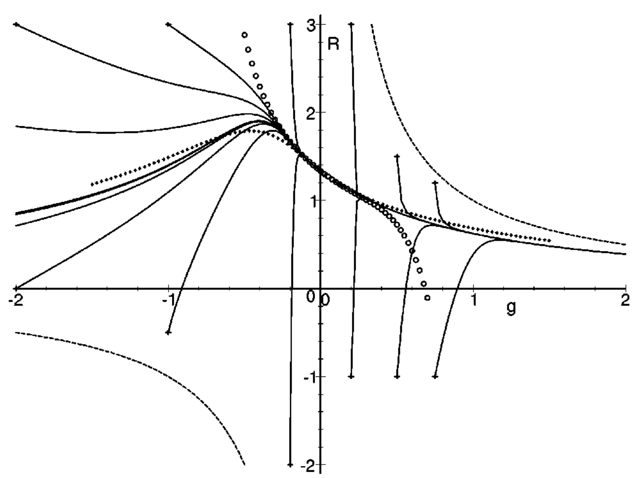

FIG. 1. Solid lines: numerical integration with various initial points (crosses). Two poorly resolved lines correspond to the initial conditions $(-100,0)$ and $(-100,3)$. Circles: Taylor expansion to the fifth order. Dots: the analytical approximation of Ref. [1]. Dash: boundaries of the domain of attraction.

favor of numerics by constructing solutions with the initial conditions within the domain A_- at large negative values of g. Then, for moderate g, these solutions will be close to the relevant one. For example, if the initial conditions of Uribe and Piña [$P=(-2,0)$ and $(-2,3)$; see the figure in Ref. [2]] are placed at $g=-100$ [$P=(-100,0)$ and $(-100,3)$], then at $g=-2$ the difference between these solutions is less than 1%. Yet another (analytical) possibility is to use the expansion in γ, as was suggested in Ref. [1], Eq. (16). This can be justified by noticing that the coefficients of the Taylor expansion [see Eq.(3) in Ref. [2]] are analytical (polynomial) functions of γ. Figure 1 demonstrates that the first approximation already provides a reasonable global result for both positive and negative g.

Thus Eq. (1) indeed defines the physical solution relevant to the viscosity factor for all values of g. The peculiarities of the numerical construction of this solution are due to the essentially singular points of irrelevant solutions, while the same points are the points of regularity of the relevant solution. As a final comment on this point, the presence of essential singularities is by no means a pathology of the model considered, but is very coherent with the presence of the invariant manifold; thus the example may be typical of other cases where the invariant manifold is defined as a solution to a differential equation.

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[1] I. V. Karlin, G. Dukek, and T. F. Nonnenmacher, Phys. Rev. E **55**, 1573 (1997).

[2] F. J. Uribe and E. Piña, preceding paper, Phys. Rev. E **57**, 3672 (1998).